AP Calculus AB — Homework 0211 SOLUTIONS
Dr. Paul L. Bailey — Tuesday, February 11, 2025 — Name:

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Problem 1. Let

$$f(x) = \frac{x}{1+x^2}.$$

Find the slope of the line tangent to the graph of f at the point $(2, \frac{2}{5})$.

Solution. We have seen that $f'(x) = \dfrac{1-x^2}{(1+x^2)^2}$. The slope is $f'(2) = \dfrac{-3}{25}$. □

Problem 2 (Thomas §4.1 # 59)**.** The function

$$V(x) = x(10-2x)(16-2x) \quad \text{for} \quad 0 < x < 5$$

models the volume of a box.

(a) Find the extreme values of V.

(b) Interpret any values found in part **(a)** in terms of volume of the box.

Solution. Compute that $V(x) = 4x^3 - 52x^2 + 160x = 4(x^3 - 13x^2 + 40x) = a^3x(5-x)(8-x)$. This can be viewed as a box obtained from a rectangle with side lengths $5a$ and $8a$, cutting out a square from each corner of length ax, and folding up the resulting tabs. Here, $a = \sqrt[3]{4}$.

So $V'(x) = 4(3x^2 - 26x + 40) = 4(3x-20)(x-2)$. Solve $V'(x) = 0$ and get $x = 2$ or $x = \frac{20}{3}$. However, $\frac{20}{3}$ is not in the domain of V, so the unique maximum is at $x = 2$ with $V(2) = 144$. Then values $x = 0$ represents a box with zero height, and $x = 5$ represents a box with height $5a$ but with a base of zero area. □

Problem 3 (Thomas §4.1 # 66)**.** If an even function $f(x)$ has a local maximum at $x = c > 0$, can anything be said about the value of f at $x = -c$? Justify your answer.

Solution. We have $f'(-c) = -f'(c)$. That is, the derivative of an even function is odd. To see this, realize that the graph of f has reflective symmetry across the y-axis. When the tangent line is reflected, its slope becomes negative. □

Problem 4 (Thomas §4.1 # 67)**.** If an odd function $g(x)$ has a local maximum at $x = c > 0$, can anything be said about the value of g at $x = -c$? Justify your answer.

Solution. We have $f'(-c) = f'(c)$. That is, the derivative of an odd function is even. To see this, realize that the graph of f has $180°$ rotational symmetry around the origin. When the tangent line is rotated, its slope does not change. □

Problem 5 (Thomas §4.1 # 69)**.** Consider a generic cubic function

$$f(x) = ax^3 + bx^2 + cx + d.$$

(a) Show that f can have 0, 1, or 2 critical points. Give examples and graphs to support your argument.

(b) How many local extreme values can f have?

Solution. Since f is a polynomial, it is differentiable on $\mathbb{R}$, so its critical points are exactly the points where its derivative is zero. We have $f'(x) = 3ax^2 + 2bx + c$. This is quadratic with discriminant $\Delta = 4b^2 - 12ac$. Thus

- if $\Delta > 0$, f' has exactly two distinct real zeros;
- if $\Delta = 0$, f' has exactly one real zero;
- if $\Delta < 0$, f' has no real zeros.

In other words,

- $b^2 > 3ac \Rightarrow f$ has exactly two critical points;
- $b^2 = 3ac \Rightarrow f'$ has exactly one critical point;
- $b^2 < 3ac \Rightarrow f'$ has no critical points.

In the first case, f has a unique local minimum and a unique local maximum. In the other two cases, it is globally monotonic. □

Problem 6. Compute

$$\int_0^1 x^2 \tan(x^3)\, dx.$$

Solution. Let $u = x^3$ so that $du = 3x^2$. Then

$$\begin{aligned}\int_0^1 x^2 \tan(x^3)\, dx &= \int_{x=0}^{x=1} \tan(u)\, du \\ &= \Big[\ln(\sec(u))\Big]_{x=0}^{x=1} \\ &= \Big[\ln(\sec(x^3))\Big]_0^1 \\ &= \ln(\sec(1)) - \ln(\sec(0)) \\ &= \ln(\sec(1)).\end{aligned}$$

□

Problem 7 (Thomas §3.6 # 30)**.** Consider the equation

$$x + \sin y = xy.$$

Use implicit differentiation to find dy/dx.

Solution. Taking $\frac{d}{dx}$ of both sides yields

$$1 + \cos(y)\frac{dy}{dx} = y + x\frac{dy}{dx}.$$

Solving for $\frac{dy}{dx}$ gives us

$$\frac{dy}{dx} = \frac{\cos(y) - x}{y - 1}.$$

□

Problem 8 (Re: Thomas §3.6 # 30)**.** Consider the equation

$$y + \sin x = xy.$$

(a) Solve for y so that y is a function of x. Let $f(x) = y$.

(b) Graph your function on a graphing calculator, and sketch the graph.

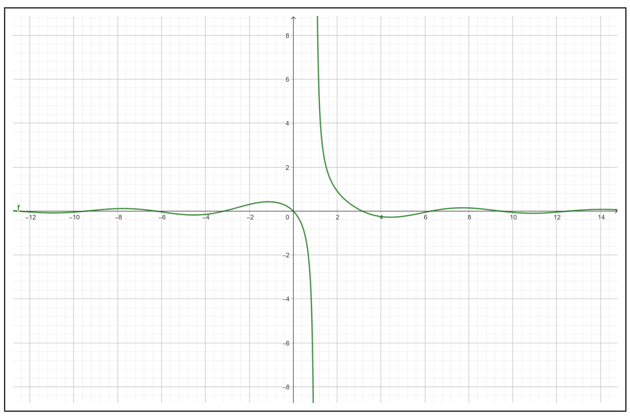
(c) What is the domain of f?

(d) Where does the equation $y + \sin x = xy$ implicitly define y as a function of x?

(e) Where does the equation $x + \sin y = xy$ implicitly define x as a function of y?

Solution. Solving to y and setting $f(x) = y$ gives us

$$f(x) = \frac{\sin x}{x-1}, \quad \text{so} \quad f'(x) = \frac{\cos(x)(x-1) - \sin(x)}{(x-1)^2}.$$



The domain of f is $\mathbb{R} \setminus \{1\}$. That is, y is a function of x on $\mathbb{R} \setminus \{1\}$. There are many branches of inverse. □

Problem 9. Compute

$$\lim_{h \to 0} \frac{\sin(a+h) - \sin a}{h},$$

where $a = \pi/3$.

Solution. If $f(x) = \sin x$, the $f'(x) = \cos(x)$, and We have

$$\lim_{h \to 0} \frac{\sin(a+h) - \sin a}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a) = \cos(a).$$

□

Problem 10. Let

$$f(x) = x^4 - 32x.$$

Find the range of f.

Solution. We have $f'(x) = 4x^3 - 32$. Setting this to zero and solving for x gives $x = 2$. The sign chart for f' tells us that f has a unique local extremum at $x = 2$, and this is a minimum. Sine $f(2) = -48$, the range must be $[-48, \infty)$. □